Group Theory 13

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Alternating Groups:- (A_n)

Alternating groups are the group of even permutations of a finite set.

A_n represents the alternating group of degree n

A_n is a subgroup of S_n.

Simple Group:- It is a group whose only normal subgroups are the trivial subgroups of order one and itself

$A_1 \rightarrow (1)$

$A_2 \rightarrow (1)$

$A_3 \rightarrow (1), (1\ 2\ 3), (1\ 3\ 2)$

- A_n is generated by 3-cycles.
- A_n is abelian iff $n \leq 3$ and simple iff $n = 3$ or $n \geq 5$
- $A_4 \supset \{(1), (12)(34), (13)(24), (14)(23)\}$ is normal

  $(12)(34)(13)(24) \neq (13)(24)(12)(34)$

  $A_4 \downarrow$ not abelian and not simple
- A_5 is simple.

Lemma:- If $H \triangleleft A_n$ and $n \geq 5$ then, if H contains a 3-cycle then $H = A_n$

Proof:- If $(a\ b\ c)$ and $(1\ 2\ 3)$ disjoint

$$(a\ b\ c)(1\ 2\ 3)(a\ b\ c)^{-1} = (1\ 2\ 3)$$

If not disjoint, then we can write it ~~as~~ two 3-cycles of A_5. So they will be conjugate. $\Rightarrow$ All 3-cycles are conjugates

As A_n is generated by 3-cycles we get $H = A_n$ as H contains all 3-cycles

Lemma :- A_5 is simple

Proof :- $(a_1\, a_2\, a_3)$, $(b_1\, b_2\, b_3) \in A_5$ are 3-cycles

$a_4 \; a_5$ $b_4 \; b_5$

We can find a permutation σ such that $\sigma(a_i) = b_i \;\; \forall i \in \{1, \ldots, 5\}$

$\sigma \in A_5$ or $S_5 \setminus A_5$

$$\sigma (a_1 \; a_2 \; a_3) \sigma^{-1} = (b_1 \; b_2 \; b_3)$$

So $(a_1\; a_2\; a_3)$ and $(b_1\; b_2\; b_3)$ are conjugates as $\sigma \in A_5$

There are 2-conjugacy classes of 5-cycles of A_5 where each class has 12 elements.

Theorem :- A_n is simple for all $n \geq 5$

Q> Prove that A_n is a subgroup of S_n

Ans :- $e \in A_n$,

$b, a \in A_n$, $\quad ab \in A_n$

Q> Prove that A_n is a normal subgroup of S_n

Ans :- $[S_n : A_n] = \mathrm{Ord}(S_n)/\mathrm{Ord}(A_n) = 2$ Hence normal

Lemma :- A_n is generated by 3-cycles

Proof :- $(a\; b\; c) \in A_n$ as $(a\; c)(a\; b) = (a\; b\; c)$

$x \in A_n$ can be written as product of even number of transpositions

$$(a_1 a_2 a_3)(a_4 a_5 a_6) \cdots (a_{3k-2}\; a_{3k-1}\; a_{3k}) \in A_n$$

$$(a_1 a_3)(a_1 a_2) \;-\;-\;-\;-\; (a_{3k-2}\; a_{3k})(a_{3k-2}\; a_{3k-1}) \in A_n$$

B = group generated by 3-cycles

$\Rightarrow B \leq A$

$(a\; b)(a\; c) = (a\; c\; b) \in A_n$ as well

$\Rightarrow$ D $\leq$ ''

$(a\ b)(a\ c) = (a\ c\ b) \in A_n$ as well

$(a_1\ a_2\ \ldots\ a_k) \to k$ odd $\in A_n$

$(a_1\ a_k)(a_1\ a_{k-1})\ \ldots\ (a_1\ a_2)$

$(a_1\ a_{k-1}\ a_k)(\ldots)\ \ldots \in B$

So $B \geq A_n$

So $B = A$